

The classification of $\frac{3}{2}$ -transitive permutation groups and $\frac{1}{2}$ -transitive linear groups

Martin W. Liebeck, Cheryl E. Praeger and Jan Saxl

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Abstract

A linear group $G \leq GL(V)$, where V is a finite vector space, is called $\frac{1}{2}$ -transitive if all the G -orbits on the set of nonzero vectors have the same size. We complete the classification of all the $\frac{1}{2}$ -transitive linear groups. As a consequence we complete the determination of the finite $\frac{3}{2}$ -transitive permutation groups – the transitive groups for which a point-stabilizer has all its nontrivial orbits of the same size. We also determine the $(k + \frac{1}{2})$ -transitive groups for integers $k \geq 2$.

1 Introduction

The concept of a finite $\frac{3}{2}$ -transitive permutation group – a non-regular transitive group in which all the nontrivial orbits of a point-stabilizer have equal size – was introduced by Wielandt in his book [16, §10]. Examples are 2-transitive groups and Frobenius groups: for the former, a point-stabilizer has just one nontrivial orbit, and for the latter, every nontrivial orbit of a point-stabilizer is regular. Further examples are provided by normal subgroups of 2-transitive groups; indeed, one of the reasons for Wielandt's definition was that normal subgroups of 2-transitive groups are necessarily $\frac{3}{2}$ -transitive.

Wielandt proved that any $\frac{3}{2}$ -transitive group is either primitive or a Frobenius group ([16, Theorem 10.4]). Following this, a substantial study of $\frac{3}{2}$ -transitive groups was undertaken by Passman in [13, 14], in particular completely determining the soluble examples. More recent steps towards the classification of the primitive $\frac{3}{2}$ -transitive groups were taken in [3] and [8]. In [3] it was proved that primitive $\frac{3}{2}$ -transitive groups are either affine or almost simple, and the almost simple examples were determined. For the affine case, consider an affine group $T(V)G \leq AGL(V)$, where V is a finite vector space, $T(V)$ is the group of translations, and $G \leq GL(V)$; this group is $\frac{3}{2}$ -transitive if and only if the linear group G is $\frac{1}{2}$ -transitive – that is, all

Table 1: Orbit sizes of $\frac{1}{2}$ -transitive groups in Theorem 1(ii),(iii)

p^d	$ G $	orbit size on $V^\#$	number of orbits
11^2	600	120	1
19^2	360	120	3
	1080	360	1
29^2	240	120	7
	1680	840	1
13^4	3360	1680	17

the orbits of G on the set $V^\#$ of nonzero vectors have the same size. The $\frac{1}{2}$ -transitive linear groups of order divisible by p (the characteristic of the field over which V is defined) were determined in [8, Theorem 6].

The main result of this paper completes the classification of $\frac{1}{2}$ -transitive linear groups. In the statement, by a *semiregular* group, we mean a permutation group all of whose orbits are regular.

Theorem 1 *Let $G \leq GL(V) = GL_d(p)$ (p prime) be an insoluble p' -group, and suppose G is $\frac{1}{2}$ -transitive on $V^\#$. Then one of the following holds:*

- (i) G is semiregular on $V^\#$;
- (ii) $d = 2$, $p = 11, 19$ or 29 , and $SL_2(5) \triangleleft G \leq GL_2(p)$;
- (iii) $d = 4$, $p = 13$, and $SL_2(5) \triangleleft G \leq \Gamma L_2(p^2) \leq GL_4(p)$.

In (ii) and (iii), the non-semiregular possibilities for G are given in Table 1.

Remarks 1. In conclusion (i) of the theorem, the corresponding affine permutation group $T(V)G$ (acting on V) is a Frobenius group, and G is a Frobenius complement (see Proposition 2.1 for the structure of these).

2. In conclusion (ii), $\mathbb{F}_p^* R$ acts transitively on $V^\#$, where $R = SL_2(5)$ and \mathbb{F}_p^* is the group of scalars in $GL(V)$, and $G = Z_0 R$ for some $Z_0 \leq \mathbb{F}_p^*$. Here $G \triangleleft \mathbb{F}_p^* R$ (hence is $\frac{1}{2}$ -transitive, since in general, a normal subgroup of a transitive group is $\frac{1}{2}$ -transitive).

3. The $\frac{1}{2}$ -transitive group G in part (iii) is more interesting. Here $G = (Z_0 R).2 \leq \Gamma L_2(13^2)$, where $R = SL_2(5)$ and Z_0 is a subgroup of $\mathbb{F}_{13^2}^*$ of order 28, and $G \cap GL_2(13^2) = Z_0 R$ has orbits on 1-spaces of sizes 20, 30, 60, 60.

Combining Theorem 1 with the soluble case in [13, 14] and the p -modular case in [8, Theorem 6], we have the following classification of $\frac{1}{2}$ -transitive linear groups. In the statement, for q an odd prime power, $S_0(q)$ is the subgroup of $GL_2(q)$ of order $4(q-1)$ consisting of all monomial matrices of determinant ± 1 .

Corollary 2 *If $G \leq GL(V) = GL_d(p)$ is $\frac{1}{2}$ -transitive on V^\sharp , then one of the following holds:*

- (i) G is transitive on V^\sharp ;
- (ii) $G \leq \Gamma L_1(p^d)$;
- (iii) G is a Frobenius complement acting semiregularly on V^\sharp ;
- (iv) $G = S_0(p^{d/2})$ with p odd;
- (v) G is soluble and $p^d = 3^2, 5^2, 7^2, 11^2, 17^2$ or 3^4 ;
- (vi) $SL_2(5) \triangleleft G \leq \Gamma L_2(p^{d/2})$, where $p^{d/2} = 9, 11, 19, 29$ or 169 .

The classification of $\frac{3}{2}$ -transitive permutation groups follows immediately from this result and those in [3]. For completeness, we state it here.

Corollary 3 *Let X be a $\frac{3}{2}$ -transitive permutation group of degree n . Then one of the following holds:*

- (i) X is 2-transitive;
- (ii) X is a Frobenius group;
- (iii) X is affine: $X = T(V)G \leq AGL(V)$, where $G \leq GL(V)$ is a $\frac{1}{2}$ -transitive linear group, given by Corollary 2;
- (iv) X is almost simple: either
 - (a) $n = 21$, $X = A_7$ or S_7 acting on the set of pairs in $\{1, \dots, 7\}$, or
 - (b) $n = \frac{1}{2}q(q-1)$ where $q = 2^f \geq 8$, and either $G = PSL_2(q)$, or $G = P\Gamma L_2(q)$ with f prime.

Turning to higher transitivity, recall (again from [16]) that for a positive integer k , a permutation group is $(k + \frac{1}{2})$ -transitive if it is k -transitive and the stabilizer of k points has orbits of equal size on the remaining points. For $k \geq 2$ such groups are of course 2-transitive so belong to the known list of such groups. Nevertheless, their classification has some interesting features and we record this in the following result.

Proposition 4 *Let $k \geq 2$ be an integer, and let X be a $(k + \frac{1}{2})$ -transitive permutation group of degree $n \geq k + 1$. Then one of the following holds:*

- (i) X is $(k + 1)$ -transitive;
- (ii) X is sharply k -transitive;

(iii) $k = 3$ and $X = P\Gamma L_2(2^p)$ with p an odd prime, of degree $2^p + 1$;

(iv) $k = 2$ and one of:

$L_2(q) \triangleleft X \leq P\Gamma L_2(q)$ of degree $q + 1$;

$X = Sz(q)$, a Suzuki group of degree $q^2 + 1$;

$X = A\Gamma L_1(2^p)$ with p prime, of degree 2^p .

Remarks 1. The sharply k -transitive groups were classified by Jordan for $k \geq 4$ and by Zassenhaus for $k = 2$ or 3 ; see [6, §7.6].

2. In conclusion (iv), the groups $Sz(q)$ and $A\Gamma L_1(2^p)$ are Zassenhaus groups – that is, 2-transitive groups in which all 3-point stabilizers are trivial (so that all orbits of a 2-point stabilizer are regular). The groups X with socle $L_2(q)$ are all $\frac{5}{2}$ -transitive, being normal subgroups of the 3-transitive group $P\Gamma L_2(q)$; some are 3-transitive, some are Zassenhaus groups, and some are neither.

The paper consists of two further sections, one proving Theorem 1, and the other Proposition 4.

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2 Proof of Theorem 1

Throughout the proof, we shall use the following well-known result about the structure of Frobenius complements, due to Zassenhaus.

Proposition 2.1 ([15, Theorem 18.6]) *Let G be a Frobenius complement.*

(i) *The Sylow subgroups of G are cyclic or generalized quaternion.*

(ii) *If G is insoluble, then it has a subgroup of index 1 or 2 of the form $SL_2(5) \times Z$, where Z is a group of order coprime to 30, all of whose Sylow subgroups are cyclic.*

The following result is important in our inductive proof of Theorem 1.

Proposition 2.2 *Let $R = SL_2(5)$, let $p > 5$ be a prime, and let V be a nontrivial absolutely irreducible $\mathbb{F}_q R$ -module, where $q = p^a$. Regard R as a subgroup of $GL(V)$, and let G be a group such that $R \triangleleft G \leq \Gamma L(V)$.*

(i) *If R is semiregular on V^\sharp , then $\dim V = 2$.*

- (ii) Suppose $\dim V = 2$ and G has no regular orbit on the set $P_1(V)$ of 1-spaces in V . Then either $q \in \{p, p^2\}$ with $p \leq 61$, or $q = 7^4$.
- (iii) If $\dim V = 2$ and G is $\frac{1}{2}$ -transitive but not semiregular on V^\sharp , then $q = 11, 19, 29$ or 169 . Conversely, for each of these values of q there are examples of $\frac{1}{2}$ -transitive, non-semiregular groups G , and they are as in Table 1 of Theorem 1.

Proof. (i) The irreducible R -modules and their Brauer characters can be found in [5], and have dimensions 2, 3, 4, 5 or 6. For those of dimension 3 or 5, the acting group is $R/Z(R) \cong A_5$, and involutions fix nonzero vectors; and for those of dimension 4 or 6, elements of order 3 fix vectors.

(ii) Let $\dim V = 2$, and suppose G has no regular orbit on $P_1(V)$. Assume for a contradiction that q is not as in the conclusion of (ii). In particular, $q > 61$ (recall that $p > 5$).

Write $\bar{R} = R/Z(R) \cong A_5$ and $\bar{G} = G/(G \cap \mathbb{F}_q^*)$. Now $N_{PGL(V)}(\bar{R}) = \bar{R}$, so it follows that $\bar{G} = \bar{R}\langle\sigma\rangle$ for some $\sigma \in P\Gamma L(V)$ (possibly trivial). Note that if $p \equiv \pm 2 \pmod{5}$ then $\mathbb{F}_{p^2} \subseteq \mathbb{F}_q$.

Consider the action of $\bar{R} \cong A_5$ on $P_1(V)$. As A_5 has 31 nontrivial cyclic subgroups, and each of these fixes at most two 1-spaces, it follows that \bar{R} has at least $(q - 62)/60$ regular orbits on $P_1(V)$. Since $q > 61$, \bar{R} has a regular orbit, and so $\bar{G} \neq \bar{R}$ by our assumption.

Let r be the order of the element σ modulo \bar{R} (so that $\mathbb{F}_{p^r} \subseteq \mathbb{F}_q$). If there is a regular \bar{R} -orbit Δ_0 on $P_1(V)$ that is not fixed by σ^i for any i with $1 \leq i \leq r - 1$, then $\bar{G}_{\Delta_0} = \bar{R}$ and so $\bar{G}_{\langle v \rangle} = 1$ for $\langle v \rangle \in \Delta_0$ and G has a regular orbit on $P_1(V)$, a contradiction. Hence $r > 1$, and for each regular \bar{R} -orbit Δ , there is a subgroup $\langle \sigma^{i(\Delta)} \rangle$, of prime order modulo \bar{R} , which fixes Δ setwise. Moreover, for $\langle v \rangle \in \Delta$, there exists $x \in \bar{R}$ such that $x\sigma^{i(\Delta)}$ fixes $\langle v \rangle$. Since there are at least $q - 62$ elements of $P_1(V)$ in regular \bar{R} -orbits, it follows that

$$|\bigcup \text{fix}_{P_1(V)}(x\sigma^j)| \geq q - 62, \quad (1)$$

where the union is over all $x \in \bar{R}$ and all j dividing r with r/j prime. Let $s = r/j$ for such j , and let $x \in \bar{R}$. If $(x\sigma^j)^s \neq 1$ then $(x\sigma^j)^s \in \bar{R}$ fixes at most two 1-spaces, and so $|\text{fix}(x\sigma^j)| \leq 2$; and if $(x\sigma^j)^s = 1$, then $x\sigma^j$ is $PGL(V)$ -conjugate to a field automorphism of order s , and $|\text{fix}(x\sigma^j)| = q^{1/s} + 1$. Hence (1) implies that

$$60 \sum_{s|r, s \text{ prime}} (q^{1/s} + 1) \geq q - 62. \quad (2)$$

Recall that $p > 5$ and $\mathbb{F}_{p^r} \subseteq \mathbb{F}_q$.

Suppose that $6|r$. The terms in the sum on the left hand side of (2) with $s \geq 5$ add to at most $r(q^{1/5} + 1)$, which is easily seen to be less than $q^{1/2} + 1$. Hence (2) gives

$$2(q^{1/2} + 1) + (q^{1/3} + 1) \geq \frac{q - 62}{60}.$$

Putting $y = q^{1/6}$ this yields $120y^3 + 60y^2 + 242 \geq y^6$, which is false for $y \geq 7$. Similarly, when $\text{hcf}(r, 6) = 1$ or 3 , we find that (2) fails. Consequently $\text{hcf}(r, 6) = 2$, and (2) gives $2(q^{1/2} + 1) \geq (q - 62)/60$, which implies that $q^{1/2} \leq 121$. Hence (as $p > 5$ and $q = p^a$ with a even), either $q = p^2$ or $q = 7^4$ or 11^4 . Then further use of (2) gives $p \leq 61$ in the former case, and also shows that $q \neq 11^4$. But now we have shown that q is as in (ii), contrary to assumption. This completes the proof.

(iii) Suppose G is $\frac{1}{2}$ -transitive but not semiregular on V^\sharp . If G has a regular orbit on $P_1(V)$, then it has a regular orbit on V^\sharp , which is not possible by the assumption in the previous sentence. Hence q must be as in the conclusion of part (ii). For these values of q , we use Magma [4] to construct $R \cong SL_2(5)$ in $SL_2(q)$, and for all subgroups of $\Gamma L_2(q)$ normalizing R , compute whether they are $\frac{1}{2}$ -transitive and non-semiregular. We find that such groups exist precisely when q is 11, 19, 29 or 169, and the examples are as in Table 1. ■

Note that part (ii) of the proposition follows from [11, Theorem 2.2] in the case where R is \mathbb{F}_p -irreducible on V . We shall need the more general case proved above.

We now embark on the proof of Theorem 1. Suppose that G is a minimal counterexample. That is,

- $G \leq GL_d(p) = GL(V)$ is an insoluble, $\frac{1}{2}$ -transitive p' -group,
- G is not semiregular on V^\sharp , and G is not as in (ii) or (iii) of the theorem, and
- G is minimal subject to these conditions.

Observe that since G is $\frac{1}{2}$ -transitive and not semiregular, it cannot have a regular orbit on V .

The affine permutation group $VG \leq AGL(V)$ is $\frac{3}{2}$ -transitive on V and not a Frobenius group, hence is primitive by [16, Theorem 10.4]. It follows that G is irreducible on V .

By [14, Theorem 1.1], G acts primitively as a linear group on V . Choose $q = p^k$ maximal such that $G \leq \Gamma L_n(q) \leq GL_d(p)$, where $d = nk$. Write $V = V_n(q)$, $G_0 = G \cap GL_n(q)$, $K = \mathbb{F}_q$ and $Z = G_0 \cap K^*$, the group of scalars in G_0 . Since G is insoluble, $n \geq 2$. Also G_0 is absolutely irreducible on V (see [8, Lemma 12.1]), so $Z = Z(G_0)$.

Lemma 2.3 *Let N be a normal subgroup of G with $N \leq G_0$ and $N \not\leq Z$, and let U be an irreducible KN -submodule of V . Then the following hold:*

- (i) N acts faithfully and absolutely irreducibly on U ;
- (ii) N is not cyclic;
- (iii) G_U acts $\frac{1}{2}$ -transitively on U^\sharp ;

(iv) if $(G_U)^U$ is insoluble and not semiregular, and $(N^{(\infty)}, |U|) \neq (SL_2(5), q^2)$ with $q \in \{11, 19, 29, 169\}$, then $U = V$.

Proof. As G is primitive on V , Clifford's theorem implies that $V \downarrow N$ is homogeneous, so that $V \downarrow N = U \oplus U_2 \oplus \cdots \oplus U_r$ with each $U_i \cong U$. Hence N is faithful on U ; it is also absolutely irreducible, as in the proof of [8, Lemma 12.2]. Hence (i) holds, and (ii) follows.

To see (iii), let $v \in U^\sharp$, $n \in N$ and $g \in G_v$. Then $vng = vgn' = vn'$ for some $n' \in N$. Hence $\{vn : n \in N\}$ is invariant under G_v . As U is irreducible under N , $\{vn : n \in N\}$ spans U , and hence G_v stabilises U . Therefore

$$|G : G_v| = |G : G_U| \cdot |G_U : G_v|.$$

As G is $\frac{1}{2}$ -transitive this is independent of $v \in U^\sharp$, and hence G_U is $\frac{1}{2}$ -transitive on U^\sharp , as in (iii).

Finally, (iv) follows by the minimality of G . ■

By [14, Theorem A], $O_r(G_0)$ is cyclic for each odd prime r , and hence is central by Lemma 2.3(ii). Consequently $F(G_0) = ZE$ where $E = O_2(G_0)$. Moreover [14, Theorem A] also shows that $\Phi(E)$ is cyclic, hence contained in Z , and $|E/\Phi(E)| \leq 2^8$.

Now let $F^*(G_0) = ZER_1 \cdots R_k$, a commuting product with each R_i quasisimple (possibly $k = 0$).

Lemma 2.4 *We have $k \geq 1$.*

Proof. Suppose $k = 0$, and write $N = F^*(G_0) = ZE$. Since $V \downarrow G$ is primitive, every characteristic abelian subgroup of E is cyclic, so E is a 2-group of symplectic type. By a result of Philip Hall ([2, 23.9]), we have $E = E_1 \circ F$ where E_1 is either 1 or extraspecial, and F is cyclic, dihedral, semidihedral or generalised quaternion; in the latter three cases, $|F| \geq 16$. Since $N = F^*(G_0)$ we have $C_{G_0}(N) \leq N$ and $G_0/C_{G_0}(N) \leq \text{Aut}(N)$. Hence $\text{Aut}(N)$ must be insoluble, and it follows that $|E_1/\Phi(E_1)| \geq 2^4$.

Now E has a characteristic subgroup $E_0 = E_1 \circ L$, where $L = C_4$ if 4 divides $|F|$ and $L = 1$ otherwise. Then $E_0 \triangleleft G$. Let U be an irreducible KE_0 -submodule of V . By Lemma 2.3, E_0 is faithful on U and G_U is $\frac{1}{2}$ -transitive on U^\sharp . Write $H = (G_U)^U$.

Assume that H is soluble. As H is $\frac{1}{2}$ -transitive on U^\sharp , it is therefore given by [14, Theorem B], which implies that one of the following holds:

- (a) H is a Frobenius complement;
- (b) $H \leq \Gamma L_1(q^u)$, where $|U| = q^u$;
- (c) $H \leq GL_2(q^u)$ with $|U| = q^{2u}$, and H consists of all monomial matrices of determinant ± 1 ;

(d) $|U| = p^2$ with $p \in \{3, 5, 7, 11, 17\}$, or $|U| = 3^4$.

In all cases except the last one in (d), it follows (using Proposition 2.1(i) for (a)) that $|E_0/\Phi(E_0)| \leq 2^2$, which is a contradiction. In the exceptional case $|U| = 3^4$ and $|E_0/\Phi(E_0)| = 2^4$. But in this case any $3'$ -subgroup of $\text{Aut}(N)$ is soluble, and hence G_0 is soluble, again a contradiction.

Hence H is insoluble. As H is not a Frobenius complement by Proposition 2.1(ii), it is not semiregular on U^\sharp , and so Lemma 2.3(iv) implies that $U = V$. Hence E_0 is irreducible on V and so F is cyclic and $N = ZE = ZE_0$. We have $|E_0/\Phi(E_0)| \leq 2^8$ by [14, Theorem A], and hence $|E_0/\Phi(E_0)| = 2^{2m}$ with $m = 2, 3$ or 4 .

Case $m = 4$. Suppose first that $m = 4$, so $E_1 = 2^{1+8}$ and $\dim V = 16$. By [14, Lemmas 2.6, 2.10] we have $E_1 = E_0$, so that $|Z|_2 = 2$ and $G_0 \leq Z \circ 2^{1+8}.O_8^\epsilon(2)$ ($\epsilon = \pm$). Also [14, Lemma 2.4] gives $(p^2 - 1)_2 \geq 2^4$, hence $p \geq 7$, and the proof of [14, Lemma 2.12] gives $|G/N| \geq q^8/2^9$. Since $G/N \leq O_8^\epsilon(2)$, it follows that $q = 7$. Hence G/N is an insoluble $7'$ -subgroup of $O_8^\epsilon(2)$ of order greater than $7^8/2^9$. Using [5], we see that such a subgroup is contained in one of the following subgroups of $O_8^\epsilon(2)$:

$$2^6.O_6^-(2), 2^{1+8}.(S_3 \times S_5) \ (\epsilon = -) \\ S_3 \times O_6^-(2), 2^6.(S_6 \times 2), 2^6.(S_5 \times S_3), (S_5 \times S_5).2 \ (\epsilon = +)$$

We now consider elements of order 3 in G . These are elements t_k lying in subgroups $O_2^-(2)^k$ of $O_8^\epsilon(2)$ for $1 \leq k \leq 4$ and acting on the 16-dimensional space V as a tensor product of k diagonal matrices (ω, ω^{-1}) with an identity matrix of dimension 2^{4-k} , where $\omega \in K^*$ is a primitive cube root of 1; there are also scalar multiples ωt_k if Z contains ωI . We compute the action of t_k on V and also the class of the image of t_k in $O_8^\epsilon(2)$ in Atlas notation, as follows:

k	action of t_k on V	Atlas notation
1	$(\omega^{(8)}, \omega^{-1(8)})$	$3A \ (\epsilon = -), 3A \ (\epsilon = +)$
2	$(1^{(8)}, \omega^{(4)}, \omega^{-1(4)})$	$3B \ (\epsilon = -), 3E \ (\epsilon = +)$
3	$(1^{(4)}, \omega^{(6)}, \omega^{-1(6)})$	$3C \ (\epsilon = -), 3D \ (\epsilon = +)$
4	$(1^{(6)}, \omega^{(5)}, \omega^{-1(5)})$	$- \ (\epsilon = -), 3BC \ (\epsilon = +)$

Hence every element of order 3 in G has fixed point space on V of dimension at most 8. Considering the above subgroups of $O_8^\epsilon(2)$, we compute that the total number of elements of order 3 in G is less than 2^{20} . If G contains an element of order 3 fixing a nonzero vector in V , then as G is $\frac{1}{2}$ -transitive, every nonzero vector is fixed by some element of G of order 3. Hence V is the union of the subspaces $C_V(t)$ over $t \in G$ of order 3, so that

$$|V| \leq \sum_{t \in G, |t|=3} |C_V(t)|. \quad (3)$$

This yields $7^{16} < 2^{20} \cdot 7^8$, which is false.

It follows that G contains no element of order 3 fixing a nonzero vector. So every element of order 3 in G/N is conjugate to t_1 .

We now complete the argument by considering involutions in G . Now G certainly contains involutions which fix nonzero vectors, so arguing as above we have

$$|V| \leq \sum_{t \in G, |t|=2} |C_V(t)|. \quad (4)$$

The group G/N is an insoluble $7'$ -subgroup of $O_8^\epsilon(2)$, all of whose elements of order 3 are conjugates of t_1 . Using Magma [4], we compute that there are 206 such subgroups if $\epsilon = +$, and 59 if $\epsilon = -$. For each of these possibilities for G/N we compute the list of involutions of G and their fixed point space dimensions. All possibilities contradict (4). For example, when $\epsilon = -$ the largest possibility for G has 188 involutions with fixed space of dimension 12; 74886 with dimension 8; and 188 with dimension 4. Hence (4) gives

$$7^{16} \leq 188 \cdot (7^{12} + 7^4) + 74886 \cdot 7^8,$$

which is false. This completes the proof for $m = 4$.

Case $m = 3$. Now suppose $m = 3$, so that $\dim V = 8$. This case is handled along similar lines to the previous one. By [14, Lemma 2.9], either $|Z|_2 = 2$ and $G_0/N \leq O_6^\epsilon(2)$, or 4 divides $|Z|$ and G contains a field automorphism of order 2 (so that q is a square), and $G_0/N \leq Sp_6(2)$. As G_0 is insoluble, its order is divisible by 2 and 3, so $p \geq 5$. Also each non-central involution in G_0 fixes a nonzero vector.

Assume now that 7 divides $|G|$. If 7 divides $|G/G_0|$ then $q \geq 5^7$ and we easily obtain a contradiction using (4); so 7 divides $|G_0|$. Elements of order 7 in G_0 act on V as $(1^2, \omega, \omega^2, \dots, \omega^6)$ where ω is a 7th root of 1 in the algebraic closure of \mathbb{F}_q (since they are rational in $O_6^+(2)$). In particular they fix nonzero vectors, so $\frac{1}{2}$ -transitivity implies

$$|V| \leq \sum_{t \in G, |t|=7} |C_V(t)|. \quad (5)$$

The number of elements of order 7 in $Sp_6(2)$ is 207360, and hence the number in G_0 is at most $(q-1, 7) \cdot 2^6 \cdot 207360$. Each fixes at most q^2 vectors, so (5) gives

$$q^8 \leq (q-1, 7) \cdot 2^6 \cdot 207360 \cdot q^2,$$

which implies that $q \leq 13$. Hence $q = 5, 11$ or 13 (not 7 as G_0 is a p' -group). As q is prime, by the first observation in this case, we have $|Z|_2 = 2$ and $G/N \leq O_6^+(2)$. But then the number of elements of order 7 in G is at most $2^6 \cdot 5760$, so (5) forces $q = 5$. So G/N is an insoluble $5'$ -subgroup of $O_6^+(2)$, and now we use Magma to see that such a group G is not $\frac{1}{2}$ -transitive on the nonzero vectors of $V = V_8(5)$.

Therefore 7 does not divide $|G|$. It follows that G_0/N is contained in one of the following subgroups of $Sp_6(2)$:

$$O_6^-(2), S_6 \times S_3, 2^5.S_6.$$

As G_0 is insoluble and a p' -group, we have $p \geq 7$. We now consider elements of order 3 in G . These are conjugate to elements t_k ($1 \leq k \leq 3$) lying in subgroups $(O_2^-(2))^k$ of $Sp_6(2)$, and acting on V as follows:

$$\begin{aligned} t_1 &: (\omega^{(4)}, \omega^{-1(4)}), \\ t_2 &: (1^4, \omega^{(2)}, \omega^{-1(2)}), \\ t_3 &: (1^2, \omega^{(3)}, \omega^{-1(3)}). \end{aligned}$$

Suppose G has an element of order 3 which fixes nonzero vectors in V , so that (3) holds. We argue as in the previous case that q is not a cube, so 3 does not divide $|G/G_0|$. In $O_6^-(2)$, the numbers of elements conjugate to t_1, t_2, t_3 are 240, 480, 80 respectively. Hence, if $G_0/N \leq O_6^-(2)$ then (3) gives

$$q^8 \leq 2^4 \cdot 480q^4 + 2^6 \cdot 80q^2 + 2^3 \cdot 240q^4 + 2^5 \cdot 480q^2 + 2^7 \cdot 80q^3$$

where the last three terms are only present if 3 divides $|Z|$. This gives $q = 7$. Similarly $q = 7$ is the only possibility if G_0/N is contained in $S_6 \times S_3$ or $2^5.S_6$. But now we compute using Magma that such groups G are not $\frac{1}{2}$ -transitive on the nonzero vectors of $V = V_8(7)$.

Thus all elements of order 3 in G are fixed point free on $V^\#$, and hence G_0/N is an insoluble $7'$ -subgroup of $Sp_6(2)$, all of whose elements of order 3 are conjugate to t_1 . We compute that there are 10 such subgroups, and for each of them, (4) implies that $q = 7$ is the only possibility: for example, the largest possible G_0 has 60 (resp. 3526, 60) involutions with fixed point spaces on V of dimension 6 (resp. 4, 2), so (4) yields

$$q^8 \leq 60q^6 + 3526q^4 + 60q^2,$$

hence $q = 7$. Finally, we compute that none of the possible subgroups G is $\frac{1}{2}$ -transitive on the nonzero vectors of $V = V_8(7)$.

Case $m = 2$. Now suppose $m = 2$, so that $\dim V = 4$. Then G_0/N is an insoluble subgroup of $Sp_4(2)$, so is isomorphic to S_6, A_6, S_5 or A_5 .

Assume that G_0/N is A_6 or S_6 . Then 4 divides $|Z|$ (so divides $q - 1$). Elements of G_0 of order 3 are conjugate to t_k ($k = 1, 2$) lying in $Sp_2(2)^k$; and t_1 acts on V as $(\omega^{(2)}, \omega^{-1(2)})$, t_2 as $(1^2, \omega, \omega^{-1})$. By assumption G_0 contains elements in both classes, so (3) yields

$$q^4 \leq 2^4 \cdot 40q^2 + 2 \cdot 2^4 \cdot 40q + 2 \cdot 2^2 \cdot 40q^2,$$

where the last two terms are present only if 3 divides $|Z|$ (hence also $q - 1$). Since 4 divides $q - 1$, we conclude that $q = 13$ or 17 in this case.

Now assume G_0/N is A_5 or S_5 . As G is a p' -group, $p \geq 7$. We compute that G_0 has at most 230 involutions, so (4) gives $q^4 \leq 230q^2$, whence $q \leq 13$.

Thus in all cases, we have $q = 7, 11, 13$ or 17. We now compute that none of the possibilities for G is $\frac{1}{2}$ -transitive on the nonzero vectors of $V = V_4(q)$. This completes the proof of the lemma. \blacksquare

Lemma 2.5 *Either $|E/\Phi(E)| \leq 2^2$, or $|E/\Phi(E)| = 2^4$ and $p = 3$.*

Proof. The result is trivial if $E \leq Z$, so suppose is not the case. Let $N = ZE \triangleleft G$, and let U be an irreducible KN -submodule of V . By Lemma 2.3, N is faithful on U and G_U is $\frac{1}{2}$ -transitive on U^\sharp . Write $H = (G_U)^U$.

Assume first that H is insoluble. Now H is not semiregular on U^\sharp (as it is not a Frobenius complement by Proposition 2.1, having $N \cong N^U$ as a normal subgroup), so Lemma 2.3(iv) implies that $U = V$. But then $N = ZE$ is irreducible on V , which forces $k = 0$, contrary to Lemma 2.4.

Hence H is soluble. As it is $\frac{1}{2}$ -transitive on U^\sharp , it is therefore given by [14, Theorem B]; the list is given under (a)-(d) in the proof of Lemma 2.4. In all cases except the last one in (d), it follows that $|E/\Phi(E)| \leq 2^2$; in the exceptional case $|U| = 3^4$ and $|E/\Phi(E)| = 2^4$. Hence the conclusion of the lemma holds. ■

Lemma 2.6 *If $R_i \triangleleft G$, then $R_i = SL_2(5)$ and $V \downarrow R_i = U^l$, a direct sum of l copies of an irreducible KR_i -submodule U of dimension 2.*

Proof. Suppose $R := R_i \triangleleft G$. By Lemma 2.3, $V \downarrow R = U^l$ with U irreducible and $(G_U)^U$ $\frac{1}{2}$ -transitive. If $(R, \dim U) = (SL_2(5), 2)$ then the conclusion holds, so suppose this is not the case. If R^U is semiregular then R is a Frobenius complement, so $R \cong SL_2(5)$; but then $\dim V$ must be 2 by Proposition 2.2(i), which we have assumed not to be the case. Therefore R^U is not semiregular, and so $U = V$ by Lemma 2.3(iv). In particular $F^*(G_0) = ZR$.

At this point we wish to apply [11, Theorem 2.2]: this states that, with specified exceptions, any p' -subgroup of $GL_d(p)$ that has a normal irreducible quasisimple subgroup, has a regular orbit on vectors. In order to apply this, we need to establish that our quasisimple normal subgroup R of G acts irreducibly on V , regarded as an $\mathbb{F}_p R$ -module. To see this, we go back to the proof of Lemma 2.3, letting $N := R \triangleleft G$. Taking U' to be an irreducible $\mathbb{F}_p R$ -submodule of V , that proof shows that R is faithful on U' , and that $G_{U'}$ is $\frac{1}{2}$ -transitive on U' . Hence by the minimality of G , either $U' = V$ (which is the conclusion we want), or $G_{U'}^{U'}$ is semiregular or as in (ii) or (iii) of Theorem 1. In the semiregular case, Proposition 2.1 implies that $R = SL_2(5)$ and U' is a 2-dimensional R -module over some extension K of \mathbb{F}_p , and this holds in (ii) and (iii) of Theorem 1 as well. However this can only happen if $\dim_K V = 2$, contradicting our assumption that $(R, \dim U) \neq (SL_2(5), 2)$. Hence $U' = V$, as desired.

Now we apply [11, Theorem 2.2] which determines all the possibilities for G not having a regular orbit on V ; these are

- (1) the case with $R = A_c$ ($c < p$) and V the deleted permutation module of dimension $c - 1$, and
- (2) the cases listed in Table 2.

Table 2: Groups in case (2) of the proof of Lemma 2.6

G/Z	n	q	$G_v \leq$	m
A_5	3	11	C_2	3
S_5	4	7	C_2	3
S_6	5	7	C_2	5
$A_{6.2}$	4	7	C_3	2
A_6	3	19, 31	C_2, C_2	5, 3
A_7	4	11	C_3	7
$L_2(7)$	3	11	C_2	3
$L_2(7).2$	3	25	C_2	3
$U_3(3).2$	7	5	C_2	7
$U_3(3).2$	6	5	S_3	4
$U_4(2)$	4	7	—	—
$U_4(2)$	5	7, 13, 19	S_4, V_4, C_2	5, 5, 5
$U_4(2).2$	6	7, 11, 13	D_{12}, V_4, C_2	5, 5, 5
$U_4(2)$	4	13, 19, 31, 37	$[18], [9], C_3, C_2$	4, 2, 2, 3
$U_4(3).2$	6	13, 19, 31, 37	$W(B_3), S_3 \times C_2, V_4, C_2$	5, 5, 5, 5
$U_5(2)$	10	7	V_4	3
$Sp_6(2)$	7	11, 13, 17, 19	C_2^3, V_4, C_2, C_2	7, 7, 7, 7
$\Omega_8^+(2)$	8	11, 13, 17, 19, 23	$W(B_3), S_4, S_3, V_4, C_2$	7, 7, 7, 7, 7
J_2	6	11	S_3	4

Case (1) In this case $G = Z_0H$ where Z_0 is a group of scalars and $H = A_c$ or S_c , and $V = \{(\alpha_1, \dots, \alpha_c) \in \mathbb{F}_p^c : \sum \alpha_i = 0\}$. If $v_1 = (1, -1, 0, \dots, 0)$ and $v_2 = (1, 1, -2, 0, \dots, 0)$, one checks that the sizes of the G -orbits containing v_1 and v_2 are $\frac{c(c-1)|Z_0|}{(2, |Z_0|)}$ and $3|Z_0|\binom{c}{3}$ respectively. These are not equal for any $c \geq 5$, contradicting $\frac{1}{2}$ -transitivity.

Case (2) In the case where $G/Z = U_4(2)$ and $(n, q) = (4, 7)$, G has two orbits on 1-spaces of sizes 40 and 360 (see [12]), and so cannot be $\frac{1}{2}$ -transitive on $V^\#$. In each other case in Table 1, [11, Theorem 2.2] gives the existence of a vector v with stabiliser G_v contained in a subgroup as indicated in column 4 of the table; and examination of the corresponding Brauer character of G of degree n in [5] gives the existence of another vector u with stabiliser G_u containing an element of order m , as indicated in column 5. It follows in all cases that G is not $\frac{1}{2}$ -transitive. ■

Lemma 2.7 *We have $k = 1$.*

Proof. Suppose $k > 1$. Assume first that $R_i \triangleleft G$ for all i . Then $N := R_1 R_2 \triangleleft G$; moreover N is not a Frobenius complement by Proposition 2.1, so is not semiregular on $V^\#$, and hence Lemma 2.3(iv) shows that N is irreducible on V . Now Lemma 2.6

implies that

$$N = R_1 R_2 = SL_2(5) \otimes SL_2(5) \leq G \leq \Gamma L_4(q).$$

Let $V = U \otimes W$ be a tensor decomposition preserved by N , with $\dim U = \dim W = 2$. If $q \neq p$ or p^2 with $p \leq 61$, and also $q \neq 7^4$, then Proposition 2.2 shows that the group induced by G/Z on 1-spaces in U has a regular orbit, and the same for W . Pick $\langle u \rangle$ and $\langle w \rangle$ in such orbits ($u \in U, w \in W$). Then $G_{\langle u \otimes w \rangle} \leq Z$ and so $G_{u \otimes w} = 1$. Hence G has a regular orbit on V^\sharp , a contradiction. And if $q = p, p^2$ or 7^4 , then

$$G \leq Z \cdot (SL_2(5) \otimes SL_2(5)).a = Z \cdot R_1 R_2.a \leq \Gamma L_4(q),$$

where a divides 4. Here $G_0 = Z \cdot R_1 R_2$. Let u_1, u_2 be a basis of U and w_1, w_2 a basis of W . Writing matrices relative to these bases, define $R_2^T = \{A^T : A \in R_2\}$. Then by [8, Lemma 4.3], for the vector $v = u_1 \otimes w_1 + u_2 \otimes w_2$ we have

$$(G_0)_v = \{B \otimes B^{-T} : B \in R_1 \cap R_2^T\}. \quad (6)$$

There is only one conjugacy class of subgroups $SL_2(5)$ in $GL_2(q)$, so we can choose bases u_i, w_i such that $R_1 = R_2^T$; then for the corresponding vector v the order of $(G_0)_v$ is divisible by 60. On the other hand there are bases for which $R_1 \cap R_2^T$ has order dividing 20, giving a vector stabilizer in G of order coprime to 3. This contradicts $\frac{1}{2}$ -transitivity.

Thus not all the R_i are normal subgroups of G . Relabelling, we may therefore take it that G permutes l factors R_1, \dots, R_l transitively by conjugation, where $l > 1$. Let $N = R_1 \dots R_l$. Lemma 2.3(iv) implies that N is irreducible on V , so that $k = l$ and $F^*(G_0) = ZN$. Now [1, (3.16), (3.17)] implies that N preserves a tensor decomposition $V = V_1 \otimes \dots \otimes V_k$ with $\dim V_i$ independent of i , $N \leq \bigotimes GL(V_i)$ and $G \leq N_{\Gamma L(V)}(\bigotimes GL(V_i)) = (GL(V_1) \circ \dots \circ GL(V_k)).S_k.\langle \sigma \rangle$ with σ a field automorphism acting on all factors.

Let G_1 be the kernel of the natural map from G to S_k , so that $G_1 = G \cap B$ where $B = (GL(V_1) \circ \dots \circ GL(V_k)).\langle \sigma \rangle$. There is a map $\phi : G_1 \rightarrow P\Gamma L(V_1)$ which has image normalizing the simple irreducible group $T := R_1/Z(R_1)$.

Just as in the second paragraph of the proof of Lemma 2.6, N acts irreducibly on V , regarded as an $\mathbb{F}_p N$ -module. It follows that R_1 acts irreducibly on V_1 , regarded as an $\mathbb{F}_p R_1$ -module: for if W_1 is a proper nonzero $\mathbb{F}_p R_1$ -submodule of V_1 , then by the transitivity of G on the R_i , there is a proper nonzero $\mathbb{F}_p R_i$ submodule W_i of V_i for each i , and then $W_1 \otimes \dots \otimes W_l$ is an $\mathbb{F}_p N$ -submodule of V , contradicting the $\mathbb{F}_p N$ -irreducibility of V .

As in the proof of Lemma 2.6, this means that we can apply [11, Theorem 2.2] to the action of $G_1 \phi$ on V_1 . This shows that one of the following holds:

- (a) $G_1 \phi$ has a regular orbit on the 1-spaces of V_1 ;
- (b) T and V_1 are among the exceptions indicated in (1) and (2) of the proof of Lemma 2.6;

(c) $(T, \dim V_1) = (A_5, 2)$.

Assume first that (a) holds and (c) does not. So $G_1\phi$ has a regular orbit on 1-spaces in V_1 . Let $\langle v \rangle$ be a 1-space in such an orbit. Write also v for the corresponding vector in the other V_i , and let H be the stabiliser $(G_1)_{v \otimes \dots \otimes v}$. Then H fixes the 1-space $\langle v \rangle \otimes \dots \otimes \langle v \rangle$, so by the choice of v , we have $H \leq Z$, the group of scalars in G . Hence in fact $H = 1$. It follows that $G_{v \otimes \dots \otimes v}$ has order dividing $k!$. Also, assuming $R_i \not\cong SL_2(r)$, there is an involution $r_i \in R_i \setminus Z$ fixing a nonzero vector $u_i \in V_i$, and hence we see that $G_{u_1 \otimes \dots \otimes u_k}$ has order divisible by 2^k . However 2^k does not divide $k!$ so this is impossible. For $R_i \cong SL_2(r)$ we have $\dim V_i > 2$ (as we are assuming (c) does not hold), and use a similar argument with an element of order 3 fixing a vector (which can be seen to exist from the character table of $SL_2(r)$ in [7]).

Now consider case (b), where T, V_1 are as in (1) or (2) of the proof of Lemma 2.6. For T, V_1 as in Table 2 (apart from $U_4(2)$ in dimension 4), let $v, u \in V_1$ be as in the last paragraph of the proof of Lemma 2.6, and let C be the group in the fourth column of Table 2 and m the integer in the fifth. Then $(G_1)_{v \otimes \dots \otimes v}$ is isomorphic to a subgroup of C^k , so that $G_{v \otimes \dots \otimes v}$ has order dividing $|C|^k k!$. On the other hand $(G_1)_{u \otimes \dots \otimes u}$ has order divisible by m^k . Since G is $\frac{1}{2}$ -transitive, this implies that m^k divides $|C|^k k!$, which is not the case.

The remaining cases in (b) are: $T = A_c$ ($c < p$), V_1 the deleted permutation module; and $T = U_4(2)$, $V_1 = V_4(7)$. In the latter case T has two orbits on 1-spaces in V_1 with stabilizers of orders 72 and 648; so as above G has a vector stabiliser of order dividing $72^k k!$ and another of order divisible by 648^{k-1} , a contradiction. Now suppose $T = A_c$ ($c < p$) and V_1 is the deleted permutation module, which we represent as $\{(x_1, \dots, x_c) \in \mathbb{F}_p^c : \sum x_i = 0\}$. By Bertrand's Postulate (see [9]) we can choose a prime r such that $\frac{c}{2} < r < c$. Let v_1, v_2 be the following vectors in V_1 :

$$v_1 = (1^r, -r, 0^{c-r-1}), \quad v_2 = (1^{r-1}, 1-r, 0^{c-r}).$$

Then $G_{v_1 \otimes \dots \otimes v_1}$ has order divisible by r^k , while $G_{v_2 \otimes \dots \otimes v_2}$ has order dividing $m^k k!$, where $m = (r-1)!(c-r)!$ (note that $1-r \neq 1$ in \mathbb{F}_p , since $p > c$). Hence r^k divides $k!$, a contradiction.

Finally consider case (c). Here $\dim V_i = 2$ and $R_i \cong SL_2(5)$; this case requires a special argument. Since R_1 is \mathbb{F}_p -irreducible on V_1 , we must have $q = p$ or p^2 , and hence $G \leq Z \cdot (SL_2(5) \otimes \dots \otimes SL_2(5)) \cdot S_k \cdot \langle \sigma \rangle$ with σ of order 1 or 2. Write $s = \lfloor \frac{k}{2} \rfloor$. As in the argument after (6), there is a vector $v \in V_1 \otimes V_2$ whose stabilizer in $SL_2(5) \otimes SL_2(5)$ contains a diagonal copy of $SL_2(5)$. Tensoring v with the corresponding vectors in $V_3 \otimes V_4, \dots, V_{2s-1} \otimes V_{2s}$ (and a further vector in V_k if k is odd), we see that there is a vector in V with stabilizer in G of order divisible by 60^s . On the other hand there is a 1-space $\langle w \rangle$ in V_1 with stabilizer in $SL_2(5)/Z(SL_2(5))$ of order dividing 2, 3 or 5. Then $|G_{w \otimes \dots \otimes w}|$ divides $t^k k! |\sigma|$ for some $t \in \{2, 3, 5\}$. Thus $60^{\lfloor k/2 \rfloor}$ divides $t^k k! |\sigma|$. This is impossible unless k is odd, $t = 5$ and there is no 1-space in V_1 with stabilizer of order dividing 2 or 3. The latter can only hold if $q \equiv 3 \pmod{4}$.

and $q \equiv 2 \pmod 3$. This implies that $q = p$ and $\sigma = 1$, so that $60^{(k-1)/2}$ divides $5^k k!$. In particular 2^{k-1} divides $k!$, which is a contradiction for k odd. \blacksquare

We can now complete the proof of Theorem 1. By Lemmas 2.6 and 2.7, we have $F^*(G_0) = ZER_1$ where $R_1 = SL_2(5)$ and $E = O_2(G_0)$. Note that $p > 5$ since G is a p' -group, and so Lemma 2.5 shows that $|E/\Phi(E)| \leq 2^2$. Also by Lemma 2.6 we have $V \downarrow R_i = U^l$, a direct sum of l copies of an irreducible KR_i -submodule U of dimension 2.

Suppose $E \not\leq Z$, so that $E/\Phi(E) = 2^2$. Write $N = F^*(G_0)$. Proposition 2.1 shows that N is not a Frobenius complement; hence Lemma 2.3 shows that N is irreducible on V . Let W be an irreducible KE -submodule of V . By Lemma 2.3, E is faithful on W (so $\dim W = 2$) and G_W^W is a soluble $\frac{1}{2}$ -transitive group. Such groups are classified in [14, Theorem B]. From this it follows that one of the following holds:

- (a) G_W^W is a Frobenius complement (so E is generalised quaternion);
- (b) relative to some basis of W we have $G_W^W = S_0(q)$, the group of monomial 2×2 matrices of determinant ± 1 ;
- (c) $|W| = p^2$ with $p \in \{7, 11, 17\}$.

In case (c), $q = p$; also $p \neq 7, 17$ as $SL_2(5) \not\leq GL_2(p)$ for these values. Hence $V = U \otimes W = V_4(p)$ with $p = 11$, and a Magma computation shows that there is no such $\frac{1}{2}$ -transitive group G in this case.

In case (a), $G_W^W \leq Z \cdot SL_2(3) < GL_2(q)$; and in (b), $G_W^W = Z \cdot 2^2 < Z \cdot SL_2(3) \cdot 2 < GL_2(q)$. In either case it follows that $V = U \otimes W$ and $G \leq Z \cdot (SL_2(5) \otimes (SL_2(3) \cdot 2)) < GL_2(q) \otimes GL_2(q) < GL_4(q)$. Write $\bar{G} = GZ/Z$, so that $\bar{G} \leq A_5 \times S_4$.

We saw in the proof of Proposition 2.2 that at least $q - 62$ of the elements of $P_1(U)$ lie in regular orbits of A_5 . Similarly, at least $q - 32$ elements of $P_1(W)$ lie in regular orbits of S_4 . Hence if $q > 61$ then, picking $\langle u \rangle \in P_1(U)$ and $\langle w \rangle \in P_1(W)$ in regular orbits, we see that $u \otimes w$ lies in a regular orbit of G on $V^\#$. This is a contradiction, since G is $\frac{1}{2}$ -transitive but not semiregular. Hence $q \leq 61$. Now a Magma computation shows that no $\frac{1}{2}$ -transitive groups arise in cases (a) and (b) as well.

Thus we finally have $F^*(G_0) = ZR_1$ with $R_1 = SL_2(5)$ and $V \downarrow R_1 = U^l$, $\dim U = 2$. Here G/Z is A_5 or S_5 , so $l = 1$. Now Proposition 2.2(iii) shows that $q = 11, 19, 29$ or 169 and G is as in conclusion (ii) or (iii) of Theorem 1. This is our final contradiction to the assumption that G is a minimal counterexample.

This completes the proof of Theorem 1.

3 Proof of Proposition 4

Let $k \geq 2$ and suppose that X is a $(k + \frac{1}{2})$ -transitive permutation group of degree n . Assume that X is not k -transitive. We refer to [10, §2] for the list of 2-transitive groups, and to [6, §7.6] for a discussion of sharply k -transitive groups.

The proposition is trivial if X is A_n or S_n , so assume this is not the case. Then $k \leq 5$, as there are no 6-transitive groups apart from A_n and S_n . Apart from A_n and S_n , the only 5-transitive groups are the Mathieu groups M_{12} and M_{24} , and the only 4-transitive, not 5-transitive, groups are M_{11} and M_{23} . The groups M_{11} and M_{12} are sharply 4- and 5-transitive respectively; and in M_{23} , a 4-point stabilizer has orbits of size 3 and 16, so that M_{23} is not $4\frac{1}{2}$ -transitive and also M_{24} is not $5\frac{1}{2}$ -transitive. This gives the proposition for $k = 4$ or 5.

Next let $k = 3$. Then X is a 3-transitive but not 4-transitive group, hence is one of the following: $AGL_d(2)$ (degree 2^d); 2A_7 (degree 2^4); M_{11} (degree 12); M_{22} or $M_{22}.2$ (degree 22); or a 3-transitive subgroup of $P\Gamma L_2(q)$ (degree $q + 1$). The affine groups here are not $3\frac{1}{2}$ -transitive, as a 3-point stabilizer fixes a further point. Neither are M_{11} , M_{22} or $M_{22}.2$ as 3-point stabilizers have orbits of size 3, 6 or 3, 16. Finally, suppose that X is a 3-transitive subgroup of $P\Gamma L_2(q)$. There are two possible sharply 3-transitive groups here, namely $PGL_2(q)$ and a group $M(q_0^2) := L_2(q_0^2).2$ with $q = q_0^2$ and q odd, which is an extension of $L_2(q_0^2)$ by a product of a diagonal and a field automorphism. Assuming that X is not one of these, it must be the case that a 3-point stabilizer $X_{\alpha\beta\gamma} = \langle \phi \rangle$, where ϕ is a field automorphism. Since X is $3\frac{1}{2}$ -transitive, $\langle \phi \rangle$ acts semiregularly on the remaining $q - 2$ points, so any nontrivial power of ϕ must fix exactly 3 points. It follows that $q = 2^p$ with p prime, and ϕ has order p , which is the example in conclusion (iii) of Proposition 4.

Now suppose that $k = 2$. Consider first the case where X is almost simple, and let $T = \text{soc}(X)$. When T is not $L_2(q)$, $Sz(q)$ or ${}^2G_2(q)$, the arguments in [10, §3] show that a 2-point stabilizer $X_{\alpha\beta}$ has orbits of unequal sizes on the remaining points, contradicting $2\frac{1}{2}$ -transitivity. The groups with socle $L_2(q)$ are in conclusion (iv) of Proposition 4. If $T = {}^2G_2(q)$ (of degree $q^3 + 1$), then $X_{\alpha\beta}$ has order $(q - 1)f$, where $f = |X : T|$ is odd, and $X_{\alpha\beta}$ is generated by an element x of order $q - 1$ and a field automorphism of odd order f . This group has a unique involution $x^{(q-1)/2}$ which fixes $q + 1$ points. It follows that some nontrivial orbits of $X_{\alpha\beta}$ have odd size and some have even size, contrary to $2\frac{1}{2}$ -transitivity. Now consider $T = Sz(q)$, of degree $q^2 + 1$. If $X = T$ then it is a Zassenhaus group, and is in (iv) of the proposition. Otherwise, $X = \langle T, \phi \rangle$ where ϕ is a field automorphism of odd order f , say, and ϕ fixes $q_0^2 + 1$ points, where $q = q_0^f$. For suitable α, β we have $X_{\alpha\beta} = \langle x, \phi \rangle$, where x has order $q - 1$, and $\langle x \rangle$ has $q + 1$ orbits of size $q - 1$. Now ϕ fixes points in some of these orbits, so by $2\frac{1}{2}$ -transitivity it must fix a point in each of them. But $|\text{fix}(\phi)| = q_0^2 + 1 < q + 1$, which is a contradiction.

Finally, suppose X is affine (with $k = 2$). Write $X = T(V)X_0 \leq AGL(V)$, where $n = |V|$, $T(V)$ is the translation subgroup, and $X_0 \leq GL(V)$. We refer to [10, §2(B)] for the list of possibilities for the transitive linear group X_0 . If $X_0 \triangleright$

$SL_d(q)$ ($n = q^d, d \geq 2$), $Sp_d(q)'$ ($n = q^d, d \geq 4$) or $G_2(q)'$ ($n = q^6$), the arguments in [10, §4] show that for some $v \in V^\sharp$, X_{0v} has nontrivial orbits of unequal sizes. In cases (6-8) of [10, §2(B)], we have $X_0 \triangleright SL_2(5)$, $SL_2(3)$, 2^{1+4} or $SL_2(13)$, and $n \in \{3^4, 3^6, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$; in each case $n - 2$ is coprime to the order of a 2-point stabilizer X_{0v} , so it follows by $2\frac{1}{2}$ -transitivity that $X_{0v} = 1$. In other words, X must be sharply 2-transitive, as in conclusion (ii) of the proposition.

It remains to deal with the case where $X \leq A := \text{AGL}_1(q)$ ($n = q$). Here A_{01} consists of field automorphisms, so if we pick $v \in \mathbb{F}_q$ such that v lies in no proper subfield of \mathbb{F}_q , then $A_{01v} = 1$. Hence by $2\frac{1}{2}$ -transitivity, all 3-point stabilizers in X are trivial – that is, X is a Zassenhaus group. It is well known that the non-sharply 2-transitive Zassenhaus groups in the 1-dimensional affine case are just $\text{AGL}_1(2^p)$ with p prime, as in (iv) of the proposition. This is easy to see: we have $X_{01} = \langle \phi \rangle$, where ϕ is a field automorphism, and this acts semiregularly on $\mathbb{F}_q \setminus \{0, 1\}$; hence, as argued at the end of the $k = 3$ case above, $q = 2^p$ with p prime and $X = \text{AGL}_1(2^p)$, as required.

This completes the proof of Proposition 4.

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Martin W. Liebeck, Dept. of Mathematics, Imperial College, London SW7 2BZ, UK, email: m.liebeck@imperial.ac.uk

Cheryl E. Praeger, School of Mathematics and Statistics, University of Western Australia, Western Australia 6009, email: praeger@maths.uwa.edu.au

Jan Saxl, DPMMS, CMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK, email: saxl@dpmms.cam.ac.uk